model with $p = -2$, $q = 2$. Consider the log-law region in the usual coordinates, and assume that the turbulence statistics are self-similar (with $L_D \sim y$, $k \sim y^0$, $\mathcal{R}_{ij}^{(e)} \sim y^{-1}$, $\bar{\mathcal{R}}_{ij} \sim y^{-1}$). Show that a solution to Eq. (11.210) is

$$\mathcal{R}_{ij}^{(e)} = \bar{\mathcal{R}}_{ij} \bigg/ \left(1 - \frac{C_L^2 \kappa^2}{C_\mu^3}[q-1][p+q-2]\right). \tag{11.211}$$

Comment on the effects of the three models mentioned above.

11.9 Algebraic Stress and Nonlinear Viscosity Models

11.9.1 Algebraic Stress Models

By the introduction of an approximation for the transport terms, a Reynolds-stress model can be reduced to a set of algebraic equations. These equations form an *algebraic stress model* (ASM) which implicitly determines the Reynolds stresses (locally) as functions of k, ε and the mean velocity gradients. Because of the approximation involved, algebraic stress models are inherently less general and less accurate than Reynolds-stress models. But, because of their relative simplicity, they have been used as turbulence models (in conjunction with the model equations for k and ε). In addition, an algebraic stress model provides some insights into the Reynolds-stress model from which it is derived; and it can also be used to obtain a nonlinear turbulent viscosity model.

A standard modelled Reynolds-stress transport equation is

$$\mathcal{D}_{ij} \equiv \frac{\bar{\mathrm{D}}\langle u_i u_j\rangle}{\bar{\mathrm{D}}t} - \frac{\partial}{\partial x_k}\left(\frac{C_s k}{\varepsilon}\langle u_k u_\ell\rangle \frac{\partial}{\partial x_\ell}\langle u_i u_j\rangle\right) = \mathcal{P}_{ij} + \mathcal{R}_{ij} - \tfrac{2}{3}\varepsilon\delta_{ij}. \tag{11.212}$$

This is a coupled set of six partial differential equations. The terms on the right-hand side are local, algebraic functions of $\partial\langle U_i\rangle/\partial x_j$, $\langle u_i u_j\rangle$ and ε—they do not involve derivatives of the Reynolds stresses. In algebraic stress models, the transport terms $\mathcal{D}_{ij}$ (on the left-hand side of Eq. 11.212) are *approximated* by an algebraic expression, so that the entire equation becomes algebraic. Specifically, Eq. (11.212) becomes a set of six algebraic equations which implicitly determines the Reynolds stresses as functions of k, ε and the mean velocity gradients.

In some circumstances (e.g., the log-law region of high-Reynolds number fully-developed channel flow), the transport terms in Eq. (11.212) are negligible, so that (in a sense) the Reynolds stresses are in local equilibrium with the imposed mean velocity gradient. However, the complete neglect of the transport terms is inconsistent unless $\mathcal{P}/\varepsilon$ is unity, since half the trace of Eq. (11.212) is

$$\tfrac{1}{2}\mathcal{D}_{\ell\ell} = \mathcal{P} - \varepsilon. \tag{11.213}$$

Rodi (1972) introduced the more general *weak equilibrium assumption.* The Reynolds stress can be decomposed as

$$\langle u_i u_j \rangle = k \frac{\langle u_i u_j \rangle}{k} = k(2b_{ij} + \tfrac{2}{3}\delta_{ij}), \tag{11.214}$$

and so spatial and temporal variations in $\langle u_i u_j \rangle$ can be considered to be due to variations in k and b_{ij}. In the weak equilibrium assumption, the variations in $\langle u_i u_j \rangle / k$ (or equivalently in b_{ij}) are neglected, but the variations in $\langle u_i u_j \rangle$ due to those in k are retained. For the mean convection term this leads to the approximation

$$\begin{aligned} \frac{\bar{\mathrm{D}}}{\bar{\mathrm{D}}t}\langle u_i u_j \rangle &= \frac{\langle u_i u_j \rangle}{k}\frac{\bar{\mathrm{D}}k}{\bar{\mathrm{D}}t} + k\frac{\bar{\mathrm{D}}}{\bar{\mathrm{D}}t}\left(\frac{\langle u_i u_j \rangle}{k}\right) \\ &\approx \frac{\langle u_i u_j \rangle}{k}\frac{\bar{\mathrm{D}}k}{\bar{\mathrm{D}}t}. \end{aligned} \tag{11.215}$$

The same approximation applied to the entire transport term yields

$$\mathcal{D}_{ij} \approx \frac{\langle u_i u_j \rangle}{k}\frac{1}{2}\mathcal{D}_{\ell\ell} = \frac{\langle u_i u_j \rangle}{k}(\mathcal{P} - \varepsilon), \tag{11.216}$$

where the last step follows from Eq. (11.213).

The use of the weak equilibrium assumption (Eq. 11.216) in the modelled Reynolds-stress equation (Eq. 11.212) leads to the algebraic stress model

$$\frac{\langle u_i u_j \rangle}{k}(\mathcal{P} - \varepsilon) = \mathcal{P}_{ij} + \mathcal{R}_{ij} - \tfrac{2}{3}\varepsilon\delta_{ij}. \tag{11.217}$$

This comprises five independent algebraic equations (since the trace contains no information), which can be used to determine $\langle u_i u_j \rangle / k$ (or equivalently b_{ij}) in terms of k, ε and $\partial\langle U_i \rangle / \partial x_j$.

As an example of the insights that an ASM can provide, if $\mathcal{R}_{ij}$ is given by the LRR-IP model, then Eq. (11.217) can be manipulated to yield

$$b_{ij} = \frac{\tfrac{1}{2}(1 - C_2)}{C_R - 1 + \mathcal{P}/\varepsilon}\frac{(\mathcal{P}_{ij} - \tfrac{2}{3}\mathcal{P}\delta_{ij})}{\varepsilon}, \tag{11.218}$$

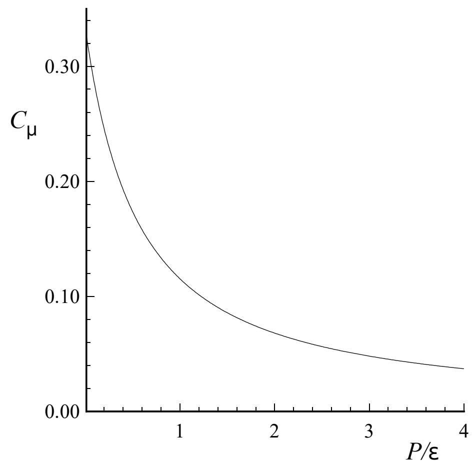

Figure 11.20: The value of C_μ as a function of $\mathcal{P}/\varepsilon$ given by the LRR-IP algebraic stress model (Eq. 11.220).

see Exercise 11.20. Consequently it may be seen that an implication of the model is that the Reynolds stress anisotropy is directly proportional to the production anisotropy.

For simple shear flow, Eq. (11.218) is readily solved to obtain the anisotropies b_{ij} as functions of $\mathcal{P}/\varepsilon$ (see Exercise 11.20): these are plotted in Fig. 11.21. For large $\mathcal{P}/\varepsilon$, $|b_{12}|$ tends to the asymptote $\sqrt{\frac{1}{6}C_2(1-C_2)} = \frac{1}{5}$, whereas the value given by the k-ε model continually increases and becomes non-realizable.

Again for simple shear flow, if the relation

$$-\langle uv \rangle = \frac{C_\mu k^2}{\varepsilon} \frac{\partial \langle U \rangle}{\partial y}, \tag{11.219}$$

is used to define C_μ, then it can be deduced (Exercise 11.31) that the ASM (Eq. 11.218) yields

$$C_\mu = \frac{\frac{2}{3}(1-C_2)(C_1 - 1 + C_2\mathcal{P}/\varepsilon)}{(C_R - 1 + \mathcal{P}/\varepsilon)^2}. \tag{11.220}$$

Consequently, as shown in Fig. 11.20, the value of C_μ implied by the LRR-IP model decreases with $\mathcal{P}/\varepsilon$, corresponding to "shear-thinning" behavior—C_μ decreases with increasing shearing, $\mathcal{S}k/\varepsilon$.

With respect to the mean flow convection, the weak equilibrium assumption (Eq. 11.215) amounts to

$$\frac{\bar{\mathrm{D}}}{\bar{\mathrm{D}}t} b_{ij} = 0, \tag{11.221}$$